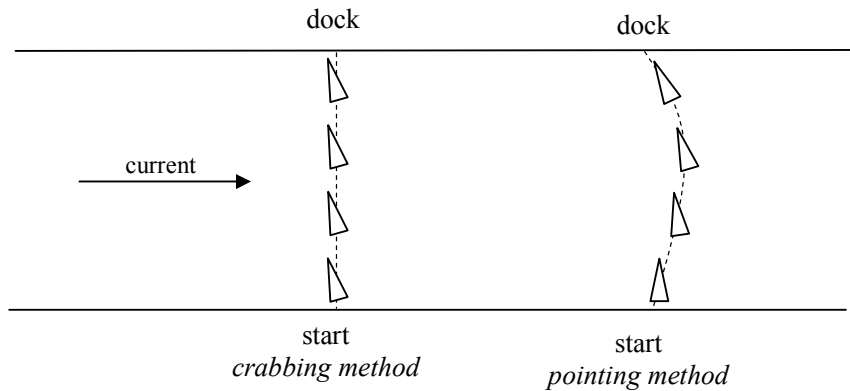


## boat time

Suppose you are anchored near the shore of a channel in which there is steady current, and you are going to run your (motor)boat at constant throttle to a dock directly across the channel on the opposite shore. There are two ways one might steer the boat to the dock:

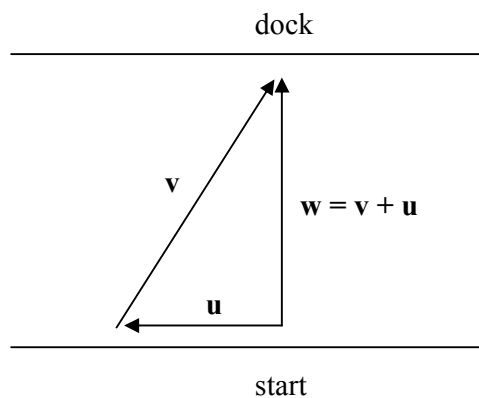
- *the crabbing method*: steer a steady course with the nose of the boat pointed somewhat upstream, so the boat maintains a fixed orientation and crabs in a straight line across the channel
- *the pointing method*: keep the nose of the boat pointed directly at the dock



Which method gets the boat to the dock faster, and by how much? (Assume the boat runs at a constant speed relative to the water, which is faster than the speed of the current relative to the shore.)

### Ilkka Mäkinen's Solution

Let's call  $\mathbf{v}$  the velocity of the boat with respect to the water and  $\mathbf{u}$  the velocity of the water with respect to the shore. In the crabbing method, the velocity  $\mathbf{w}$  of the boat with respect to the shore, which is the vector sum of  $\mathbf{v}$  and  $\mathbf{u}$ , is directed towards the dock, so we can draw the following diagram:



From the diagram it is seen that

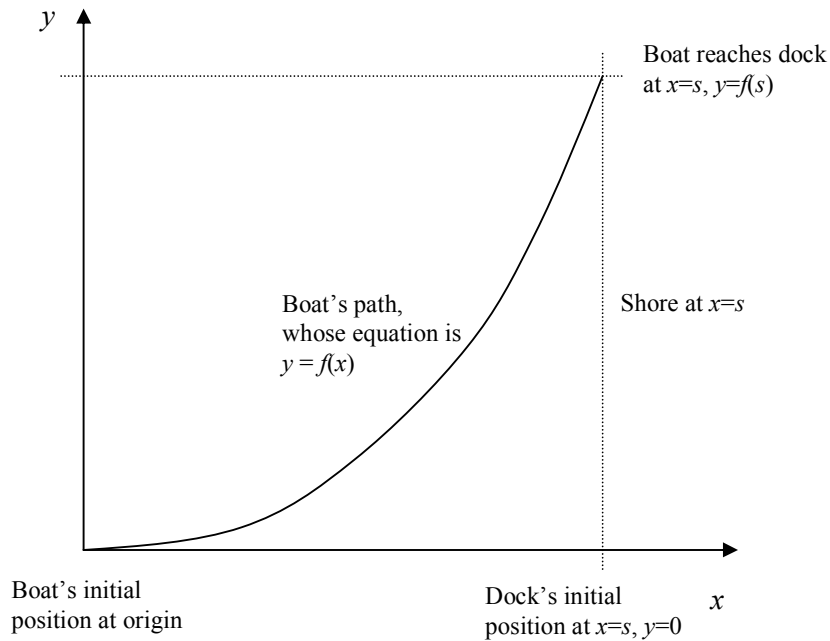
$$w = \sqrt{v^2 - u^2}$$

Thus the time required for crossing the channel is

$$t_{crab} = \frac{s}{\sqrt{v^2 - u^2}} \quad (1)$$

where  $s$  is the width of the channel.

For the pointing method, consider the situation from the frame of the water in the channel. Then it will look as if the dock were running away from the boat with a constant velocity  $\mathbf{u}$ , and the boat were chasing the dock with a velocity  $\mathbf{v}$  of constant magnitude, and whose direction is always towards the dock. After some time the boat will reach the dock. On chasing the dock, the boat will have described some curve in the frame of the water – let the equation of this curve be  $y = f(x)$ . (See the following diagram.)

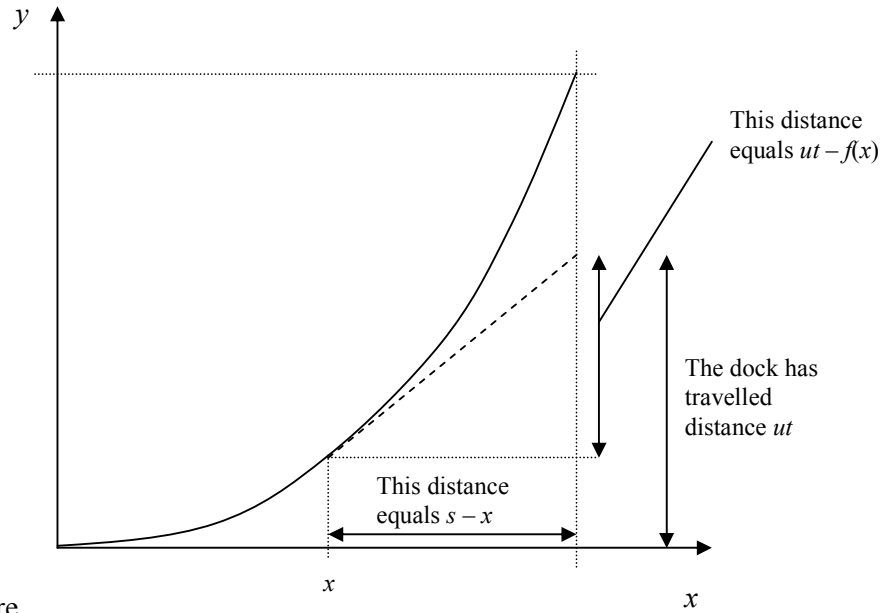


Now, let the boat's  $x$ -coordinate at some particular instant be  $x$ . Then the distance the boat has travelled so far is given by

$$\int_0^x \sqrt{1 + f'(x)^2} dx. \quad (2)$$

This must be equal to  $vt$ , where  $t$  is the time elapsed since the boat started its journey. Since the nose of the boat is always directed towards the dock, it follows that the tangent of  $f(x)$  at time  $t$  points towards the position of the dock at  $t$ . By inspecting the following diagram, it can be seen that the slope of the tangent, which is given by  $f'(x)$  is

$$f'(x) = \frac{ut - f(x)}{s - x}.$$



Therefore

$$ut = f'(x)(s - x) + f(x).$$

Solving this equation for  $t$  and setting the expression (2) equal to  $vt$ , one gets

$$\frac{u}{v} \int_0^x \sqrt{1 + f'(x)^2} dx = f'(x)(s - x) + f(x).$$

To get rid of the integral on the left, the equation can be differentiated with respect to  $x$ , giving

$$\frac{u}{v} \sqrt{1 + f'(x)^2} = f''(x)(s - x)$$

whence, by setting  $z = f'(x)$  and separating the variables,

$$\frac{dz}{\sqrt{1 + z^2}} = \frac{u}{v(s - x)} dx. \quad (3)$$

By integrating this equation, with the initial condition  $f'(0) = 0$ , one finds that

$$\ln(z + \sqrt{z^2 + 1}) = \frac{u}{v} \ln\left(\frac{s}{s - x}\right). \quad (4)$$

The initial condition comes from the fact that initially the dock is directly in front of the boat, so the boat's velocity at  $x=0$  is in the direction of the  $x$ -axis.

[The left side of (3) can be integrated by multiplying and dividing the integrand by  $1 + z/\sqrt{1 + z^2}$ . The integrand is then transformed to  $(1 + z/\sqrt{1 + z^2}) / (z + \sqrt{1 + z^2})$ , which is of the form "derivative of a function divided by the function itself". The integral of this kind of an expression is the natural logarithm.]

The left side of (4) can be recognized as the inverse hyperbolic sine of  $z$ ; therefore

$$\begin{aligned} z = f'(x) &= \sinh \left[ \frac{u}{v} \ln \left( \frac{s}{s-x} \right) \right] = \frac{1}{2} \left( \frac{s}{s-x} \right)^{u/v} - \frac{1}{2} \left( \frac{s-x}{s} \right)^{u/v} \\ &= \frac{1}{2} \left( 1 - \frac{x}{s} \right)^{-u/v} - \frac{1}{2} \left( 1 - \frac{x}{s} \right)^{u/v} \end{aligned} \quad (5)$$

[The hyperbolic sine is defined as  $\sinh x = \frac{1}{2}(e^x - e^{-x})$ . Its inverse function, which we needed to know to get Eq. (5) from Eq. (4), can be derived from the definition of the inverse function,  $f(f^{-1}(x)) = x$ , using the definition of the hyperbolic sine.]

Now Eq. (5) can be integrated, with the initial condition  $f(0) = 0$ , to find  $f(x)$ :

$$f(x) = \frac{sv}{2(u+v)} \left( 1 - \frac{x}{s} \right)^{1+(u/v)} + \frac{sv}{2(u-v)} \left( 1 - \frac{x}{s} \right)^{1-(u/v)} + \frac{svu}{v^2 - u^2}.$$

Now  $f(s)$ , the  $y$ -coordinate of the point where the boat will reach the dock, can be found from the above equation:

$$f(s) = \frac{svu}{v^2 - u^2}.$$

Incidentally,  $f(s)$  is just the distance that the dock has travelled between its initial and final positions. Since  $u$  is the velocity of the boat, the time it has taken the dock to reach  $f(s)$  is

$$t = \frac{sv}{v^2 - u^2}.$$

Because the dock and the boat meet each other at  $f(s)$ , the time in which the boat will reach the dock must be the same. Therefore the time for the pointing method is

$$t_{point} = \frac{sv}{v^2 - u^2}. \quad (6)$$

Finally, on comparing Eqs. (1) and (6), we conclude that

$$\frac{t_{point}}{t_{crab}} = \frac{1}{\sqrt{1 - u^2/v^2}}$$

i.e. the crabbing method is faster by a relative difference of  $\sqrt{1 - u^2/v^2}$ .