bursting shell

A shell flying with velocity 500 m/s bursts into three identical fragments so that the kinetic energy of the system increases 1.5 times. What maximum velocity can one of the fragments obtain?

David Peterson's Solution

The total momentum of the fragments after the shell bursts must equal the momentum of the shell before it burst. If m is the mass of the shell (so that fragments have mass $m/3$), $\vec{v}_0$ is the velocity of the shell, and $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are the velocities of the fragments, then

$$\frac{m}{3}\vec{v}_1 + \frac{m}{3}\vec{v}_2 + \frac{m}{3}\vec{v}_3 = m\vec{v}_0,$$

or,

$$\vec{v}_1 = 3\vec{v}_0 - (\vec{v}_2 + \vec{v}_3). \qquad (1)$$

From (1) it can be seen that the magnitude of $\vec{v}_1$ is maximized when $\vec{v}_1$ is in the same direction as $\vec{v}_0$ while both $\vec{v}_2$ and $\vec{v}_3$ are in the opposite direction. Since $v_0 = 500$, we have

$$v_1 = 3v_0 - (v_2 + v_3) = 1500 - (v_2 + v_3). \qquad (2)$$

It is given that the burst increses the kinetic energy of the system 1.5 times, so

$$\left(½\frac{m}{3}v_1^2\right) + \left(½\frac{m}{3}v_2^2\right) + \left(½\frac{m}{3}v_3^2\right) = 1.5\left(½mv_0^2\right),$$

or,

$$v_1^2 = 1125000 - \left(v_2^2 + v_3^2\right). \qquad (3)$$

From (3) it can be seen that v_1 is maximized when $\left(v_2^2 + v_3^2\right)$ is minimized. Since by (2) $v_2 + v_3$ is fixed at $½(1500 - v_1)$, we must have $v_2 = v_3$.[†]

Thus (2) and (3) reduce to:

$$\begin{aligned} v_1 &= 1500 - 2v_2 \\ v_1^2 &= 1125000 - 2v_2^2 \end{aligned} \qquad (4)$$

Solving (4) yields $v_1 = 0$ and $v_2 = v_3 = 750$, or $v_1 = 1000$ and $v_2 = v_3 = 250$. Thus , the maximum velocity attainable by one fragment is 1000 m/s.

[†] Given $a + b = S$, the minimum of $y = a^2 + b^2 = a^2 + (S - a)^2$ is achieved when $dy/da = 0$, yielding $a = S/2 = b$.